On Some Degree and Distance Invariants of Graphs

Ph.D Thesis
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On Some Degree and Distance Invariants of Graphs

A thesis submitted to

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In partial fulfillment
of the requirement for the degree of

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in
Mathematics

by
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Dedication

To My Parents and All Family

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Praise to be ALLAH, the Cherisher and Lord
of the World, Most gracious and Most Merciful

In the name of the Omniscient Allah, the Most Gracious, the Most Merciful, who gave me the chance to take on this task and gave me the skills I needed to complete it. I give all of my praise to the human race's blessing, the Holy Prophet Hazrat Muhammad (S.A.W). I consider myself extremely fortunate to have Dr. Faisal Nadeem as my supervisor. Without the motivating guidance and ongoing encouragement, this work would never have been completed. He advised me on my thesis, for which I am really grateful. My ebullience and desire for learning were preserved thanks to my honourable professors' inspiration and support. I acknowledge, appreciate, and respond to the support and affection of my family, without whom I would have been lost.

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Abstract

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By

Faiza Ishfaq

Due to the vast usage of networks, numerous networks are considered and widely practiced in several branches of science, i.e., in engineering, chemistry, biology, and computer networking. Networks can be described in graphs, where a vertex is affiliated with a node and the connection between them as edges.

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Chapter 1
Introduction

Leonhard Paul Euler (1707–1783), a famous Swiss geometrician who spent the majority of his life in Germany and Russia, is known as the father of graph theory [**?**]. The "Königsberg bridge" problem, named after the Königsberg city located on the canal preger, was one of Euler's most difficult problems to solve. The problem was the seven bridges of the city, which connected two islands to the mainland. Others pondered whether it was conceivable to walk across each bridge just once, despite the possibility that the beginning and ending point would not be the same. A picture of Königsberg bridge depicted in Figure **??**.

Table 1.1: This is new Table format for thesis

(Ξl)	Frequency	Set of vertices
1	$4\,(r+s+t-2)$	V_1
2	$6rst+2rs+2st+2tr-2r-2s-$ $2t+6$	V_2
4	$5rst-rs-st-tr+r+s+t-1$	V_3

About 300 years ago, when no one was known about the subject of graph theory, at that time the problem of Seven Bridges of Konigsberg was arisen. Konigsberg was a city of Germany at that time and now it is a part of Russia situated on the Pregel river. This city contains seven bridges and it was connected with two islands via these seven bridges. The people of Konigsberg always thought that is there any possibility to cross all the seven bridges in one attempt without crossing the every bridge more than once. In 1736 [**?**], the person who had solved this problem was Leonhard Euler (1707-1783). Euler declared the conclusion about the problem that there is no way or it is impossible to cross these bridges in one attempt without crossing the bridges more than once. He used the dots (vertices) to express the landmasses and lines (edges) to express the seven bridges to solve this problem in a very simple way. He not only proved that it is impossible but also gave the reason why this is so. He explained this by defining the new term valance or degree of a vertex means the number of edges connected to a

particular vertex. This notion leads to the birth of Eulerian graph. Actually this notion from Euler has opened the new area in the premises of mathematics named "Graph Theory". Graph theory is the branch of combinatorics. After 100 years, another invention made by Kirchhoff [**?**] in this field while he was working with electrical networks. Caley [**?**] and Sylvester [**?**] have proposed the properties of new class of graphs called trees. Graph theory has a close relationship with linear algebra another branch of mathematics. Another invention related to matrix theory and incidence matrix discovered by Poincare [**?**] in the premises of graph theory. The family of complete graph has been discovered by Mobious in 1840. In 1852, Gutherie developed the four color problem in theory of graphs.

Graph theory is a subject which is being applied now a days in many fields of sciences such as Computer science as networks, Chemistry as chemical graphs, Electrical Engineering as electrical circuits, Operation research as sewerage system, traffic flow, telephone lines. It is also used in optimization problem. There are several applications of graph theory which are very much helpful for mankind. Graph theory belongs to mathematics but its roots are penetrating in Economics, Biology, Statistics, Architecture, Communication, Management, Mechanical, Civil and Chemical engineering.

1.1 Application of Graph Theory in Diverse Areas

There are numerous applications in daily life where graph theory can be applied. Some of them are being mentioned here.

Traffic Signals

1.2 Elements of Basic Graph Theory

Definition 1.2.1. A graph G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge two vertices (not necessarily distinct) called its endpoints.[**?**]

Definition 1.2.2. If two vertices having an edge between them then these vertices are called **adjacent vertices** and all those edges who have a common vertex are known as

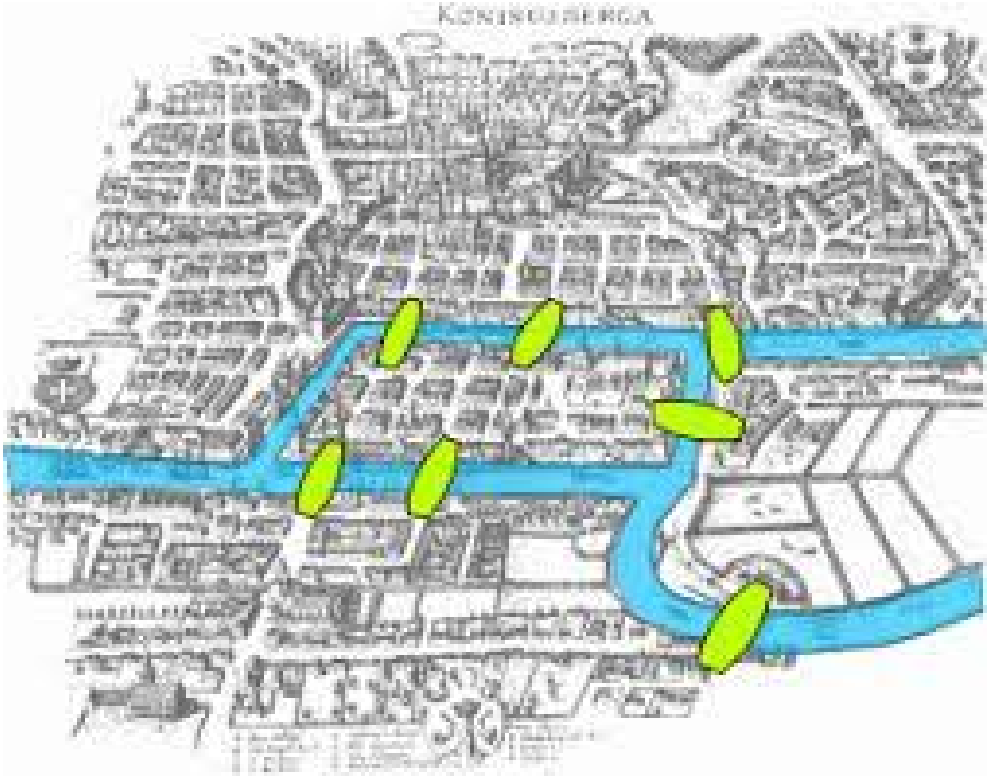

Figure 1.1: Exact and estimated comparison of Estrada index of BRE

adjacent edges.

For example in Figure 2.1, vertex p is adjacent to vertex q and r and q is adjacent to p, r and s and edges pq and pr are adjacent because they have a common vertex p between them and edges qr and qs are adjacent because q is a common vertex between them.

Definition 1.2.3. If two vertices share more than one edge, then these edges are said to be **parallel edges**.

Cycle Graph

A connected graph that is regular of degree 2 is a cycle graph. it is denoted by C_n with n vertices$(n \geq 3)$.

Chapter 2
Literature Review

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List of Publications

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